SOME LIGHT ON LITTLEWOOD-PALEY THEORY

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ABSTRACT. The purpose of this note is to correct an error in a paper of M. Cowling, G. Fendler and J.J.F. Fournier, and to give a counterexample to a conjecture of J.-L. Rubio de Francia.

Classical Littlewood–Paley theory (in the context of analysis on \mathbb{R}) deals with expressions of the form

$$\left(\sum_{I\in\mathcal{I}}\left|S_If\right|^2\right)^{1/2},$$

where \mathcal{I} is a collection of (essentially) disjoint intervals in \mathbb{R} , typically the set \mathcal{D} of dyadic intervals $\{[-2^{n+1}, -2^n], [2^n, 2^{n+1}] : n \in \mathbb{Z}\}$, and S_I is the operator of Fourier multiplication by the characteristic function of the interval I, i.e., $(S_I f)^{\hat{}} = \chi_I \hat{f}$. The classical Littlewood–Paley inequality states that, if $1 , then there exist (positive) constants <math>A_p$ and B_p such that

$$A_p \|f\|_p \le \left\| \left(\sum_{I \in \mathcal{D}} |S_I f|^2 \right)^{1/2} \right\|_p \le B_p \|f\|_p \qquad \forall f \in L^p(\mathbb{R}).$$

The collection of intervals may be replaced by other collections of intervals, but for the left hand inequality to hold, it is important that the union of the intervals be (essentially) all of \mathbb{R} ; this is not a restriction for the right hand inequality.

This has been generalized in a number of ways. J.-L. Rubio de Francia [2] proved that, if \mathcal{I} is any collection of disjoint intervals and $2 \leq p < \infty$, then an inequality

(1)
$$\left\| \left(\sum_{I \in \mathcal{D}} \left| S_I f \right|^2 \right)^{1/2} \right\|_p \le B_p \left\| f \right\|_p \qquad \forall f \in L^p(\mathbb{R})$$

still holds. He also observed that this inequality cannot hold if $1 . Indeed, if <math>\mathcal{J}$ is the collection of all intervals $\{[n, n+1] : n \in \mathbb{Z}\}$, and f_N denotes the function on \mathbb{R} whose Fourier transform \hat{f}_N is the characteristic function $\chi_{[0,N]}$, for some positive integer N, then it is straightforward to check that

$$\left\| \left(\sum_{I \in \mathcal{J}} \left| S_I f_N \right|^2 \right)^{1/2} \right\|_p = N^{1/2} \left\| f_1 \right\|_p,$$

while

$$||f_N||_p = N^{1/p'} ||f_1||_p$$

1

where p' is the dual index to p, that is, p' = p/(p-1). Thus (1) can hold only if $p \ge 2$. However, for this example, the modified Littlewood–Paley inequality

(2)
$$\left\| \left(\sum_{I \in \mathcal{D}} |S_I f|^{p'} \right)^{1/p'} \right\|_p \le B_p \|f\|_p$$

holds. Perhaps a little optimistically, Rubio de Francia [2, p. 10] conjectured that (2) might always hold. One of the aims of this paper is to provide a counterexample to this conjecture.

Shortly after Rubio de Francia's paper, Cowling, Fendler and Fournier [1] investigated variants of Littlewood–Paley theory in which mixed norms like that in (2) appear. These were used to give some examples of multipliers with some special properties. Cowling, Fendler and Fournier [1, p. 340] used the space called $D(\mathbb{R})$ of all integrable functions f on \mathbb{R} such that $\int_{n}^{n+1} f(x) dx = 0$ for all integers n. In particular, they claimed that the real interpolation space $[D(\mathbb{R}), L^{2}(\mathbb{R})]_{\theta,p}$ is the Lebesgue space $L^{p}(\mathbb{R})$, where $1/p = 1 - \theta/2$. A second purpose of this paper is to disprove this assertion; consequently all results based on this "fact" are suspect.

This paper owes much to T.-S. Quek, who observed that the interpolation theorem above would (if true) imply Rubio de Francia's conjecture.

1. A counterexample to Rubio de Francia's conjecture. Let \mathcal{I} be the family of all intervals $I_{j,n}$ of the form

$$[n+j2^{-n}, n+(j+1)2^{-n}],$$

where $n=0,1,2,\ldots$ and $0 \leq j < 2^n$. Again take the function f_N such that $\hat{f}=\chi_{[0,N]};$ then

(3)
$$||f_N||_p = N^{1/p'} ||f_1||_p.$$

Now consider one of the intervals $I_{j,n}$ above, where n < N. The absolute value of the function $S_{I_{j,n}} f_N$ is equal to the absolute value of the function $f_{2^{-n}}$ whose Fourier transform is the characteristic function $\chi_{[0,2^{-n}]}$, i.e., to the absolute value of the function $x \mapsto \sin(2^{-n}\pi x)/\pi x$ (using the Fourier transform with 2π in the exponent). Thus $|S_{I_{j,n}} f_N|$ is greater than $2^{1-n}/\pi$ on the interval of length 2^n centered at the origin. But for each n there are 2^n intervals $I_{j,n}$. Summing, we see that

$$\left(\sum_{j=0}^{2^{n}-1} \left| S_{I_{j,n}} f_{N}(x) \right|^{p'} \right)^{1/p'} \ge \frac{2^{1-n/p}}{\pi} \chi_{[-2^{n-1}, 2^{n-1}]}(x)$$

for all x in \mathbb{R} , whence

$$\left(\sum_{I \in \mathcal{I}} |S_I f_N(x)|^{p'}\right)^{1/p'} \ge \frac{2}{\pi (4|x|)^{1/p}}$$

when $1/4 \leq |x| \leq 2^N/4$, and so

$$\left\| \left(\sum_{I \in \mathcal{I}} |S_I f|^{p'} \right)^{1/p'} \right\|_p \ge \frac{2}{\pi} \left(2 \int_{1/4}^{2^N/4} \frac{1}{4x} \, \mathrm{d}x \right)^{1/p} = \frac{2}{\pi} \left(\frac{N \log 2}{2} \right)^{1/p}.$$

This inequality, together with (3), shows that Rubio's conjecture cannot hold if 1 .

2. The reason why the interpolation "theorem" is not correct. A function u lies in the real interpolation space $[D(\mathbb{R}), L^2(\mathbb{R})]_{\theta,q}$ (constructed using the J-method) if and only if it may be represented as a vector-valued integral, with values in $D(\mathbb{R}) \cap L^2(\mathbb{R})$, convergent in $D(\mathbb{R}) + L^2(\mathbb{R})$, and a fortiori convergent in $L^1(\mathbb{R}) + L^2(\mathbb{R})$:

$$u = \int_0^\infty u_t \, \frac{\mathrm{d}t}{t},$$

where

$$\left(\int_0^\infty \left(t^{-\theta} \max\{\|u_t\|_{D(\mathbb{R})}, t \|u_t\|_{L^2(\mathbb{R})}\}\right)^q \frac{\mathrm{d}t}{t}\right)^{1/q} < \infty.$$

For almost all t in \mathbb{R}^+ , u_t has to lie in $D(\mathbb{R})$, so

$$\int_{n}^{n+1} u_t(x) \, \mathrm{d}x = 0.$$

Since the map $M_n: u \mapsto \int_n^{n+1} u(x) dx$ is continuous on $L^1(\mathbb{R}) + L^2(\mathbb{R})$,

$$M_n u = \int_0^\infty M_n u_t \, \frac{\mathrm{d}t}{t} = 0,$$

so the interpolation space is a subspace of the space of all locally integrable functions on \mathbb{R} whose integrals on the intervals [n, n+1] all vanish.

References

- [1] M. Cowling, G. Fendler and J.J.F. Fournier, Variants on Littlewood–Paley theory, *Math. Annalen*, **285** (1989), 333–342.
- [2] J.-L. Rubio de Francia, A Littlewood-Paley theorem for arbitrary intervals, Rev. Mat. Iberoamericana, 1 (1985), 1-14.

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